PHYS 102 – General Physics II Midterm Exam

Duration: 150 minutes | Saturday, 16 April 2023; 15:30

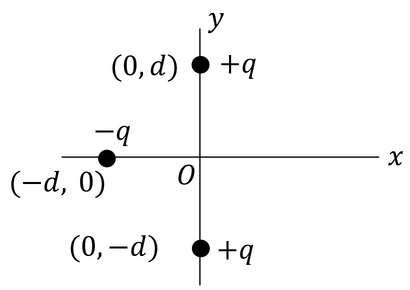
1. Three point charges are fixed on the xy-plane. Charges $+q$ are fixed at points $(0, \pm d)$, and charge $-q$ is fixed at the point $(-d, 0)$ as shown.

(a) (5 Pts.) What is the electrical force on the charge at the point $(-d, 0)$?

(b) (5 Pts.) What is the electric field $\vec{\mathbf{E}}(x)$ on the x-axis for $0 \le x < \infty$?

(c) (5 Pts.) What is the electric potential $V(x)$ on the x-axis for $0 \le x < \infty$?

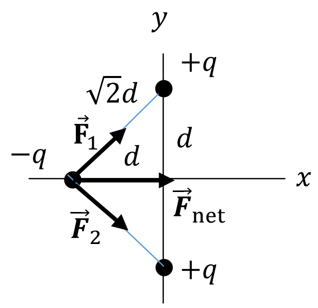
(d) (5 Pts.) What is the electric potential energy of the charge distribution?



Solution: (a)

$$|\vec{\mathbf{F}}_1| = |\vec{\mathbf{F}}_2| = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{2d^2}\right)$$

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = 2|\vec{\mathbf{F}}_1|\cos\left(\frac{\pi}{4}\right)\hat{\mathbf{i}} \quad \rightarrow \quad \vec{\mathbf{F}}_{\text{net}} = \frac{\sqrt{2}\,q^2}{8\pi\epsilon_0 d^2}\,\hat{\mathbf{i}}\,.$$

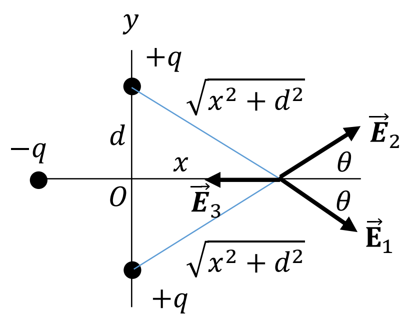


(b)

$$|\vec{\mathbf{E}}_1(x)| = |\vec{\mathbf{E}}_2(x)| = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{x^2+d^2}\right), \qquad |\vec{\mathbf{E}}_3(x)| = \frac{1}{4\pi\epsilon_0}\frac{q}{(x+d)^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+d^2}}$$

$$\vec{\mathbf{E}}(x) = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 + \vec{\mathbf{E}}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{2x}{(x^2+d^2)^{3/2}} - \frac{1}{(x+d)^2}\right]\hat{\mathbf{i}}\,.$$



(c)

$$V(x) = V_1(x) + V_2(x) + V_3(x) = \frac{q}{4\pi\epsilon_0}\left(\frac{2}{\sqrt{x^2+d^2}} - \frac{1}{x+d}\right).$$

(d)

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q^2}{2d} - \frac{q^2}{\sqrt{2}d} - \frac{q^2}{\sqrt{2}d}\right) \quad \rightarrow \quad W = \frac{q^2}{4\pi\epsilon_0 d}\left(\frac{1-2\sqrt{2}}{2}\right).$$

2. According to a crude model, the neutron consists of an inner core of positive charge surrounded by an outer shell of negative charge. Suppose that the positive chage has magnitude $+q$ and is uniformly distributed inside a sphere of radius R, and the charge $-q$ is is uniformly distributed inside the concentric shell of inner radius R and outer radius $2R$.

(a) (4 Pts.) What are the charge densities in the two regions $0 \le r < R$ and $R < r < 2R$?

(b) (8 Pts.) Find the magnitude $E(r)$ of the electric field in regions $0 \le r < R$, $R < r < 2R$, and $2R < r < \infty$.

(c) (8 Pts.) What is the potential difference between the center of the neutron and infinity?

Solution: (a)

$$\rho_1 = \frac{q}{\frac{4}{3}\pi R^3} = \frac{3q}{4\pi R^3}\,, \qquad r < R\,, \qquad \rho_2 = \frac{-q}{\frac{4}{3}\pi(2R)^3 - \frac{4}{3}\pi R^3} = \frac{-3q}{28\pi R^3}\,, \qquad R < r < 2R\,.$$

(b) Use Gauss's law $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = Q_{\text{enc}}/\epsilon_0$ where the surface is a sphere of radius r with its center at the center of the neutron.

$$4\pi r^2 E(r) = \frac{1}{\epsilon_0}\rho_1\left(\frac{4}{3}\pi r^3\right) \;\rightarrow\; E(r) = \frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\,, \qquad r < R\,.$$

$$4\pi r^2 E(r) = \frac{1}{\epsilon_0}\left[q - \rho_2\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi R^3\right)\right] \;\rightarrow\; E(r) = \frac{q}{4\pi\epsilon_0 r^2}\left(\frac{8R^3 - r^3}{7R^3}\right)\,, \qquad R < r < 2R\,.$$

$$Q_{\text{enc}} = 0 \;\rightarrow\; E(r) = 0\,, \qquad r > 2R\,.$$

(c)

$$V_{BA} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} \;\rightarrow\; V_0 = -\int_\infty^0 \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = \int_0^R E(r)dr + \int_R^{2R} E(r)dr + \int_{2R}^\infty E(r)dr\,.$$

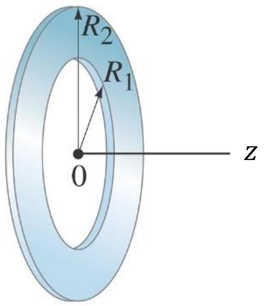
$$\int_0^R E(r)dr = \frac{q}{4\pi\epsilon_0 R^3}\int_0^R r\,dr = \frac{q}{8\pi\epsilon_0 R}$$

$$\int_R^{2R} E(r)dr = \frac{q}{4\pi\epsilon_0}\int_R^{2R}\left(\frac{8}{7r^2} - \frac{r}{7R^3}\right)dr = \frac{q}{4\pi\epsilon_0}\left(\frac{5}{14R}\right)$$

$$\int_{2R}^\infty E(r)dr = 0$$

$$V_0 = \frac{q}{4\pi\epsilon_0 R}\left(\frac{1}{2} + \frac{5}{14}\right) \;\rightarrow\; V_0 = \frac{3q}{14\pi\epsilon_0 R}\,.$$

3. A non-conducting thin annulus (a flat ring) of inner radius R_1 and outer radius R_2 is placed on the xy-plane. The annulus is uniformly charged with total charge $+Q$.



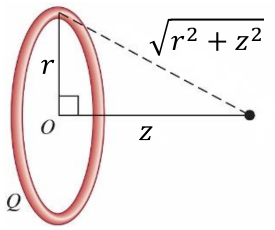
(a) (4 Pts.) Find the surface charge distribution σ on the annulus.

(b) (8 Pts.) Find the potential $V(z)$, along the z-axis.

(c) (8 Pts.) Find the electric field $\vec{\mathbf{E}}(z)$, along the z-axis.

Solution: (a)

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi(R_2^2 - R_1^2)}.$$



(b) Consider a ring of radius r with total charge Q. Electric potential on the symmetry axis of the ring is found as

$$V(z) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\sqrt{r^2+z^2}} \quad \rightarrow \quad V(z) = \frac{Q}{4\pi\epsilon_0\sqrt{r^2+z^2}}.$$

If we now think of the annulus as made up of concentric rings with radii r, and thickness dr for $R_1 < r < R_2$, potential due to one such ring can be written as

$$dV = \frac{dq}{4\pi\epsilon_0\sqrt{r^2+z^2}}, \qquad dq = \sigma\, dA = \frac{Q\, 2\pi r\, dr}{\pi(R_2^2 - R_1^2)} = \frac{2Q\, r\, dr}{(R_2^2 - R_1^2)}.$$

Integrating, we find

$$V(z) = \frac{Q}{2\pi\epsilon_0(R_2^2 - R_1^2)}\int_{R_1}^{R_2} \frac{r\, dr}{\sqrt{r^2+z^2}}.$$

The integral can be evaluated using the substitution $u^2 = r^2 + z^2$, $u\, du = r\, dr$. The result is

$$V(z) = \frac{Q}{2\pi\epsilon_0(R_2^2 - R_1^2)}\left[\sqrt{R_2^2+z^2} - \sqrt{R_1^2+z^2}\right].$$

(c)

$$E_z = -\frac{dV}{dz} \quad \rightarrow \quad E_z = \frac{Q}{2\pi\epsilon_0(R_2^2 - R_1^2)}\left[\frac{z}{\sqrt{R_1^2+z^2}} - \frac{z}{\sqrt{R_2^2+z^2}}\right].$$

$$\vec{\mathbf{E}} = E_z\,\hat{\mathbf{k}}.$$

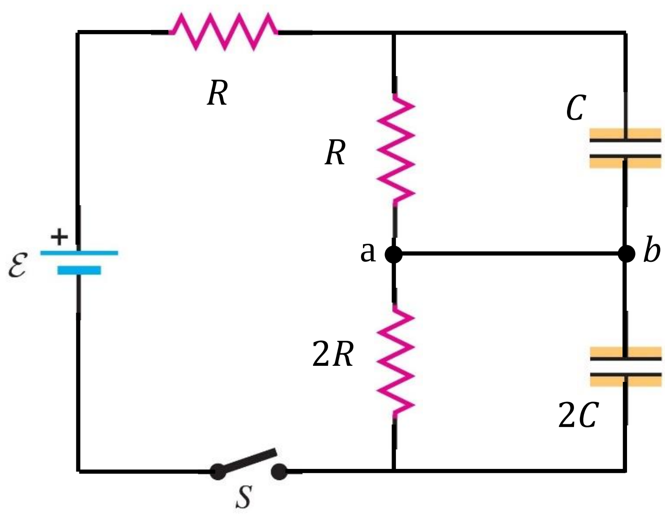
4. Consider the circuit with an ideal battery, three resistors and two uncharged capacitors shown in the figure. The switch S is closed at time $t = 0$.

(a) (5 Pts.) What is the current through the battery at time $t = 0$?

(b) (5 Pts.) What is the potential difference V_{ab} in the limit $t \to \infty$?

(c) (5 Pts.) What is the maximum charge on each capacitor?

(d) (5 Pts.) After a very long time, the switch S is opened. Which capacitor reaches half its maximum charge earlier? Why?



Solution:

(a) At time $t = 0$ uncharged capacitors behave like a short circuit and the current follows the path of least resistance. Therefore, at $t = 0$ the current through the battery is

$$i(0) = \frac{\mathcal{E}}{R}.$$

(b) Points a and b are connected by a wire. Therefore,

$$V_{\mathrm{ab}} = 0.$$

(c) The current through the single loop is

$$i(\infty) = \frac{\mathcal{E}}{4R}.$$

Potential differences actross the two capacitors are equal to the potential differences across the two resistances with whic they are connected in parallel. Therefore, maximum charge on the top capacitor is

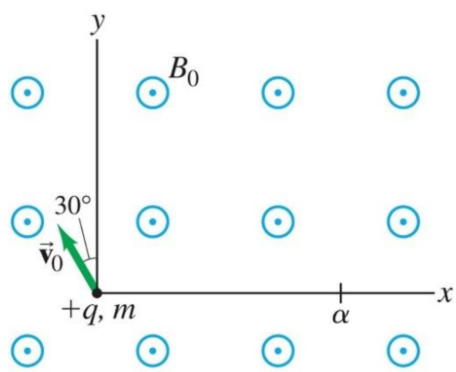
$$Q_{\mathrm{T-max}} = V_C C = V_R C = \left(\frac{\mathcal{E}}{4R}\right) RC \quad \to \quad Q_{\mathrm{T-max}} = \frac{1}{4}\mathcal{E}C,$$

while the maximum charge on the bottom capacitor is

$$Q_{\mathrm{B-max}} = V_{2C} 2C = V_{2R} 2C = \left(\frac{\mathcal{E}}{4R}\right)(2R)(2C) \quad \to \quad Q_{\mathrm{B-max}} = \mathcal{E}C.$$

(d) The top capacitor will discharge through the top resistance faster because the time constant $\tau_T = RC$ is less than that of the bottom capacitor, which will discharge through the bottom resistance with time constant $\tau_B = 4RC$.

5. A particle with charge q and mass m travels in a uniform magnetic field $\vec{\mathbf{B}} = B_0\,\hat{\mathbf{k}}$. At time $t = 0$, the particle's speed is $\vec{\mathbf{v}}_0$, and its velocity vector lies in the xy plane directed at an angle of 30° with respect to the y axis as shown in the figure. Answer the following questions in terms of q, m, v_0 and B_0.



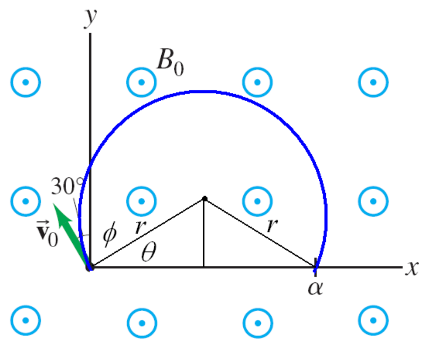
(a) (6 Pts.) What is the magnitude of the acceleration of the particle at $t = 0$?

(b) (7 Pts.) At what time t_α does the particle cross the x axis?

(c) (7 Pts.) At what point $x = \alpha$ does the particle cross the x axis?

Solution:

(a) In the magnetic field the particle will move along an arc of a circle with constant speed (i.e., uniform circular motion).



$$\vec{\mathbf{F}} = q\,\vec{\mathbf{v}}_0 \times \vec{\mathbf{B}} \;\;\rightarrow\;\; F = ma = qv_0B_0 \;\;\rightarrow\;\; a = \frac{qv_0B_0}{m}.$$

(b) For uniform circular motion, we have

$$a = \frac{v_0^2}{r} \;\rightarrow\; r = \frac{mv_0}{qB_0}.$$

From the figure, we have $\theta = 30°$, $\phi = 60°$. Therefore, the arc followed by the particle is $2/3$ full circle.

$$t_\alpha = \frac{s}{v_0}, \qquad s = \left(\frac{2}{3}\right)(2\pi r), \qquad r = \frac{mv_0}{qB_0} \;\rightarrow\; t_\alpha = \frac{4\pi m}{3qB_0}.$$

(c) From the figure,

$$\alpha = 2r\cos\left(\frac{\pi}{6}\right) \;\;\rightarrow\;\; \alpha = \sqrt{3}\,\frac{mv_0}{qB_0}.$$